Invariant norms on the p-adic Schrödinger representation

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May 25, 2021

Topological group	Coefficient field	Examples	
Real Lie groups	$\mathbb{R}$ or $\mathbb{C}$	$L^2(SL_2(\mathbb{R}))$	
p-adic groups	$\mathbb{R}$ or $\mathbb{C}$	$L^2(\mathbb{Q}_p)$	• Tate's Thesis • Smooth representations in automorphic representations
p-adic groups	$\mathbb{K}/\mathbb{Q}_p$	$C_0(\mathbb{Q}_p, \mathbb{C}_p)$	• p-adic functional analysis • p-adic L-functions • p-adic Langlands

You are here

In this talk: $\mathbb{K} = \mathbb{C}_p$ is the completion (w.r.t. $|\cdot|_p$) of an algebraic closure of $\mathbb{Q}_p$.

Fourier transform on $(\mathbb{Q}_p, +)$ over C

- C: an algebraically closed field of characteristic zero.
- The space of Schwartz functions

$$\mathcal{S}(\mathbb{Q}_p, C) = \left\{ f : \mathbb{Q}_p \to C \;\middle|\; \begin{array}{l} f \text{ is locally constant and} \\ \text{compactly supported} \end{array} \right\}.$$

  (There are plenty such functions!)
- Choose a non-trivial *smooth* character $\psi : (\mathbb{Q}_p, +) \to C^\times$.
- The Fourier transform of $f \in \mathcal{S}(\mathbb{Q}_p, C)$ is defined by

$$\widehat{f}(x) = \mathcal{F}(f)(x) = \int_{\mathbb{Q}_p} \psi(xt) f(t) \; dt.$$

- $\int_{\mathbb{Q}_p} dt$ is the *Haar functional* with $\displaystyle\int_{\mathbb{Q}_p} \mathbf{1}_{\mathbb{Z}_p}(t) \; dt = 1$.

Fourier transform on $(\mathbb{Q}_p, +)$ over $\mathbb{C}$ and $\mathbb{C}_l$

- $\widehat{\widehat{f}}(x) = f(-x)$ (if $\ker(\psi) = \mathbb{Z}_p$). In particular, $f \mapsto \widehat{f}$ is a linear isomorphism.
- For $a \in \mathbb{Q}_p$ fixed, $\widehat{f(t+a)}(x) = \psi(-ax) \cdot \widehat{f}(x)$.
- For $b \in \mathbb{Q}_p$ fixed, $\widehat{\psi(bt) \cdot f(t)}(x) = \widehat{f}(x+b)$.

<u>When $C = \mathbb{C}$:</u>

- The Fourier transform is unitary for the inner product

$$\langle f, g \rangle = \int_{\mathbb{Q}_p} f(t)\overline{g(t)}\, dt.$$

- Extends to an isometric isomorphism $\mathcal{F} : L^2(\mathbb{Q}_p) \xrightarrow{\sim} L^2(\mathbb{Q}_p)$.
- Fact: $\langle \cdot, \cdot \rangle$ is, up to a scalar, the unique inner product on $\mathcal{S}(\mathbb{Q}_p)$ invariant under translations and multiplication by smooth characters.

Fourier transform on $(\mathbb{Q}_p, +)$ over C_p

Assume $C = \mathbb{C}_p$.

- $\mathcal{S}(\mathbb{Q}_p) := \mathcal{S}(\mathbb{Q}_p, \mathbb{C}_p)$.
- No notion of an inner product with values in $\mathbb{C}_p$.
- Next best option: $\|\cdot\|_\infty$.
- $C_0(\mathbb{Q}_p)$ denotes the completion of $\mathcal{S}(\mathbb{Q}_p)$ with respect to $\|\cdot\|_\infty$.
- $\int_{\mathbb{Q}_p} dt$ does not extend to continuous functions with compact support.

$$\int_{\mathbb{Q}_p} \mathbf{1}_{p^n \mathbb{Z}_p}(t)\, dt = \frac{1}{p^n} \in \mathbb{C}_p.$$

- The Fourier transform is not continuous with respect to the sup norm.

Example

Let $n \in \mathbb{Z}$ and detnoe $f(x) = \mathbf{1}_{p^n \mathbb{Z}_p}(x)$. Then

$$\widehat{f}(x) = \frac{1}{p^n} \cdot \mathbf{1}_{p^{-n} \mathbb{Z}_p}(x).$$

Thus, $\|f\|_\infty = 1$ while $\|\widehat{f}\|_\infty = p^n$.

Fourier transform on $(\mathbb{Q}_p, +)$ over C_p

Theorem (Fresnel - de Mathan, Amice - Escassut, 1970's)

The extension of

$$\mathcal{F} : \mathcal{S}(\mathbb{Q}_p/\mathbb{Z}_p) \to \mathcal{S}(\mathbb{Z}_p)$$

to

$$\mathcal{F} : C_0(\mathbb{Q}_p/\mathbb{Z}_p) \to C(\mathbb{Z}_p)$$

is not injective.
Fresnel and de Mathan also showed: surjective.

Theorem (Discontinuity, de Shalit - O. 2017)

The graph

$$\Gamma = \{(f, \widehat{f}) \mid f \in \mathcal{S}(\mathbb{Q}_p)\} \subset C_0(\mathbb{Q}_p) \times C_0(\mathbb{Q}_p)$$

is dense. In other words, for any $g_1, g_2 \in C_0(\mathbb{Q}_p)$ and any $\epsilon > 0$ there exists $f \in \mathcal{S}(\mathbb{Q}_p)$ such that

$$\|f - g_1\|_\infty < \epsilon \quad \text{and} \quad \left\|\widehat{f} - g_2\right\|_\infty < \epsilon.$$

Fourier transform on $(\mathbb{Q}_p, +)$ over C_p

- We can reformulate the discontinuity theorem as follows.
  - Let $\|\cdot\|^\wedge$ be the Fourier norm: $\|f\|^\wedge = \|\widehat{f}\|_\infty$.
  - The discontinuity theorem $\iff$ there exists no seminorm on $\mathcal{S}(\mathbb{Q}_p)$ which is smaller than both $\|\cdot\|_\infty$ and $\|\cdot\|^\wedge$.
  - Both norms are invariant under translations and multiplication by smooth characters.

We will show:

- Put the previous two results (non-injectivity, discontinuity) under one framework.
- As a substitution to the L^2-theory:
  - A result about rigidity of the sup norm.
  - Partial results about classification of norms on $\mathcal{S}(\mathbb{Q}_p)$ that are invariant under translations and multiplication by smooth characters?

Heisenberg groups

- We fix a positive integer d.
- The standard $2d$-dimensional symplectic space: $(\mathbb{Q}_p^d \oplus \mathbb{Q}_p^d, \omega)$ where, if $w_1 = (a_1, b_1)$ and $w_2 = (a_2, b_2)$ then

$$\omega(w_1, w_2) = a_1 \cdot b_2 - a_2 \cdot b_1 = w_1 \cdot \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} w_2^t.$$

- The Heisenberg group attached to it, $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$, has underlying set

$$\left(\mathbb{Q}_p^d \oplus \mathbb{Q}_p^d\right) \times \mathbb{Q}_p$$

  and multiplication

$$[w_1, t_1] \cdot [w_2, t_2] = [w_1 + w_2, t_1 + t_2 + \frac{1}{2}\omega(w_1, w_2)].$$

- The center of $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$ is $\{[0, t] \mid t \in \mathbb{Q}_p\}$ and is equal to its commutator subgroup.
- $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$ is a totally disconnected, locally compact topological group.

The Schrödinger representation of $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$

- Let $\psi : (\mathbb{Q}_p, +) \to \mathbb{C}_p^\times$ be a non-trivial smooth character.
- $\mathcal{S}(\mathbb{Q}_p^d)$ - locally constant, compact support $f : \mathbb{Q}_p^d \to \mathbb{C}_p$.
- The Schrödinger representation $\rho_\psi : \mathcal{H}_{2d+1}(\mathbb{Q}_p) \to \mathrm{GL}(\mathcal{S}(\mathbb{Q}_p^d))$ is defined by

$$(\rho_\psi([w,t])f)(x) = \psi\left(t - \frac{1}{2}a\cdot b\right) \cdot \psi(b\cdot x) \cdot f(x+a),$$

  where $w = (a, b)$.
- Convention: $[w,t]f = \rho_\psi([w,t])f$.

Definition

A rep' V of a t.d. group G is called *smooth* if $\mathrm{Stab}_G(v)$ is open for any $v \in V$.

Theorem (Smooth Stone - von Neumann)

1. *The Schrödinger representation ρ_ψ is smooth and irreducible.*
2. *Any smooth irreducible representation of $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$ with a central character ψ is isomorphic to ρ_ψ.*

Intertwining operators

- Right action of

$$\mathrm{Sp}_{2d}(\mathbb{Q}_p) = \{g \in \mathrm{GL}_{2d}(\mathbb{Q}_p) \mid \omega(w_1 g, w_2 g) = \omega(w_1, w_2)\}$$

  on $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$ by continuous automorphisms:

$$g \in \mathrm{Sp}_{2d}(\mathbb{Q}_p) : \quad [w,t] \cdot g = [wg, t],$$

  and $[0,t] \cdot g = [0,t]$.
- For $g \in \mathrm{Sp}_{2d}(\mathbb{Q}_p)$ define a new representation $\rho_{\psi,g}$ of $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$ by

$$\rho_{\psi,g}([w,t]) = \rho_\psi([w,t] \cdot g).$$

- $\rho_{\psi,g}$ is a smooth irreducible representation with central character ψ.
- By the Stone-von Neumann theorem, there exists $T_g : \rho_\psi \xrightarrow{\sim} \rho_{\psi,g}$. Explicitly,

$$T_g([w,t]f) = ([w,t] \cdot g)\, T_g(f).$$

- T_g is called intertwining operator, is unique up to a scalar (Schur's lemma).
- $g \mapsto T_g$ is a projective representation of $\mathrm{Sp}_{2d}(\mathbb{Q}_p)$.

Remark

$g \mapsto T_g$ becomes true representation (the Weil rep') of the Metaplectic group.

Intertwining operators

Example

Let $d = 1$. The group $\mathcal{H}_3(\mathbb{Q}_p)$ can be identified with the group of matrices of the form

$$\begin{pmatrix} 1 & a & t \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

We have: $\mathrm{Sp}_2(\mathbb{Q}_p) = \mathrm{SL}_2(\mathbb{Q}_p)$. The Fourier transform is an intertwining operator for $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Indeed:

- $[(a, b), t] \cdot g = [(b, -a), t]$.
- $$\widehat{[(a, b), t]f} = [(b, -a), t]\widehat{f} \iff \begin{cases} \widehat{f(t + a)}(x) = \psi(-ax) \cdot \widehat{f}(x) \\ \widehat{\psi(bt) \cdot f(t)}(x) = \widehat{f}(x + b) \end{cases}.$$

Invariant norms and the orbit of the sup norm

- If $\|\cdot\|$ is a $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$-invariant norm on $\mathcal{S}(\mathbb{Q}_p^d)$, then $f \mapsto \|T_g(f)\|$ is again an invariant norm. Indeed,

$$\|T_g([w,t]f)\| = \|([w,t]\cdot g)T_g(f)\| = \|T_g(f)\|.$$

- A right action of $\mathrm{Sp}_{2d}(\mathbb{Q}_p)$ on the homothety classes of invariant norms.
- Can be shown: T_g fixes the homothety class of $\|\cdot\|_\infty$ if and only if

$$g \in \mathbf{P} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathrm{Sp_{2d}(\mathbb{Q}_p)} \right\}.$$

- The orbit of the sup norm is in bijection with $\mathbf{Gr} = \mathbf{P}\backslash\mathrm{Sp}_{2d}(\mathbb{Q}_p)$.

Definition

Given $\alpha = \mathbf{P}\cdot g \in \mathbf{Gr}$, we denote by $\|\cdot\|_\alpha$ the unique norm in the homothety class of $\|T_g(\cdot)\|_\infty$ for which $\left\|\mathbf{1}_{\mathbb{Z}_p^d}(x)\right\|_\alpha = 1$.

- If $\alpha, \beta \in \mathbf{Gr}$ are different, $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are not equivalent.

Simultaneous discontinuity

- Recall: $\{(f, \widehat{f}) \mid f \in \mathcal{S}(\mathbb{Q}_p\}$ is dense in $C_0(\mathbb{Q}_p) \times C_0(\mathbb{Q}_p)$.

Proposition

Let $\alpha_1 = \mathbf{P} \cdot g_1, ..., \alpha_n = \mathbf{P} \cdot g_n$, distinct in **Gr**. *Let $T_{g_1}, ..., T_{g_n}$ be corresponding intertwining operators. The following subset is dense.*

$$\left\{\left(T_{g_1}(f), ..., T_{g_n}(f)\right) \mid f \in \mathcal{S}(\mathbb{Q}_p^d)\right\} \subset C_0(\mathbb{Q}_p^d)^n.$$

Banach representations

A *unitary Banach representation* of a group G is a representation of G on a Banach space $(B, \|\cdot\|)$ over $\mathbb{C}_p$ such that $\|\cdot\|$ is G-invariant.

Definition

Let $(B, \|\cdot\|)$ be a unitary Banach representation of a group G.

1. $v \in B$ is called *topologically cyclic* if for any $w \in B$ and any $\epsilon > 0$ there exist $g_1, ..., g_n \in G$ and $\lambda_1, ..., \lambda_n \in \mathbb{C}_p$ such that
$$\left\| w - \sum_{i=1}^{n} \lambda_i \cdot g_i(v) \right\| < \epsilon.$$
2. $v \in B$ is called *strongly cyclic* if any $w \in B$ can be written as
$$w = \sum_{g \in G} \lambda_g \cdot g(v),$$
where $\lim_g \lambda_g = 0$ in $\mathbb{C}_p$.

Completions w.r.t. the norms $\|\cdot\|_\alpha$

Definition

A unitary Banach representation B of G is called *strongly irreducible* if any $0 \neq v \in B$ is strongly cyclic.

Notation: for $\alpha \in \mathbf{Gr}$, denote by $\mathcal{S}(\mathbb{Q}_p^d)_{\|\cdot\|_\alpha}$ the completion of $\mathcal{S}(\mathbb{Q}_p^d)$ w.r.t. $\|\cdot\|_\alpha$

Theorem (O. 2020)

Let $\alpha \in \mathbf{Gr}$.

- $\mathcal{S}(\mathbb{Q}_p^d)_{\|\cdot\|_\alpha}$ *is a strongly irreducible unitary Banach rep' of* $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$.
- *If* $\beta \in \mathbf{Gr}$ *and* $\beta \neq \alpha$, *then*

$$\mathcal{S}(\mathbb{Q}_p^d)_{\|\cdot\|_\alpha} \not\simeq \mathcal{S}(\mathbb{Q}_p^d)_{\|\cdot\|_\beta}.$$

A rigidity result for the completion w.r.t. $\|\cdot\|_\alpha$

Notation: $\mathrm{Hom}(B_1, B_2)$ - continuous $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$-equivariant maps.

Theorem (O. 2020)

Let $(B, \|\cdot\|)$ be a unitary Banach representation of $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$.

1. *Let $T \in \mathrm{Hom}(\mathcal{S}(\mathbb{Q}_p^d)_{\|\cdot\|_\alpha}, B)$ non-zero. Then, up to scaling the norm, T is an isometry onto its image.*
2. *Let $T \in \mathrm{Hom}(B, \mathcal{S}(\mathbb{Q}_p^d)_{\|\cdot\|_\alpha})$ non-zero. Then T is surjective and, up to scaling the norm, induces an isometry*

$$B/\ker(T) \xrightarrow{\sim} \mathcal{S}(\mathbb{Q}_p^d)_{\|\cdot\|_\alpha}.$$

Corollary

For $d = 1$, $C_0(\mathbb{Q}_p)$ is a strongly irreducible representation of $\mathcal{H}_3(\mathbb{Q}_p)$. If B (as above) is topologically irreducible, the following are equivalent

1. $\mathrm{Hom}(B, C_0(\mathbb{Q}_p)) \neq 0$.
2. $\mathrm{Hom}(C_0(\mathbb{Q}_p), B) \neq 0$.
3. B is isometrically isomorphic to $C_0(\mathbb{Q}_p)$.

Rigidity of $\|\cdot\|_\alpha$

Main Theorem (Rigidity, O. 2020)

*Let $\alpha \in$ **Gr**. Let $\|\cdot\|$ be a $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$-invariant norm on $\mathcal{S}(\mathbb{Q}_p^d)$ that is dominated by $\|\cdot\|_\alpha$ (i.e. $\|\cdot\| \leq D \cdot \|\cdot\|_\alpha$). Then $\|\cdot\| = r \cdot \|\cdot\|_\alpha$ for some $r > 0$.*

	$\mathcal{S}(\mathbb{Q}_p^d, \mathbb{C})$	$\mathcal{S}(\mathbb{Q}_p^d, \mathbb{C}_l),\ l \neq p$	$\mathcal{S}(\mathbb{Q}_p^d, \mathbb{C}_p)$
Uniqueness of invariant norms	Unique invariant unitary structure.	Unique invariant norm: $\|\cdot\|_\infty$.	Each $\|\cdot\|_\alpha$ unique among dominated norms.
Irreducibility of completion	$L^2(\mathbb{Q}_p^d)$ top' irr'	$C_0(\mathbb{Q}_p^d, \mathbb{C}_l)$ strongly irr'.	$\mathcal{S}(\mathbb{Q}_p^d)_{\|\cdot\|_\alpha}$ strongly irr'.
Smooth vectors in completion	$\mathcal{S}(\mathbb{Q}_p^d)$.	$\mathcal{S}(\mathbb{Q}_p^d)$.	$\mathcal{S}(\mathbb{Q}_p^d)$.
Stone - von Neumann theorem	$L^2(\mathbb{Q}_p^d)$ - unique irr' unitary rep' with central character ψ.	$C_0(\mathbb{Q}_p^d, \mathbb{C}_l)$ - unique cont' irr' rep' with central character ψ.	**?**

"Small norms"

- Given a finite subset $I = \{\alpha_1, ..., \alpha_n\} \subset \mathbf{Gr}$, denote

$$\|\cdot\|_I := \max(\|\cdot\|_{\alpha_1}, \|\cdot\|_{\alpha_2}, ..., \|\cdot\|_{\alpha_n}).$$

(Call $\|\cdot\|_I$ a small norm).

Proposition

Let $\|\cdot\|$ be a $\mathcal{H}_{2d+1}(\mathbb{Q}_p)$-invariant norm, dominated by $\|\cdot\|_I$ for a finite $I \subset \mathbf{Gr}$. Then there exists $J \subset I$ such that $\|\cdot\|$ is equivalent to $\|\cdot\|_J$.

Corollary

Let $I_1, I_2 \subset \mathbf{Gr}$ finite disjoint subsets. There does not exist a seminorm on $\mathcal{S}(\mathbb{Q}_p^d)$ which is smaller than both $\|\cdot\|_{I_1}$ and $\|\cdot\|_{I_2}$.

About the proofs

Two types of methods.

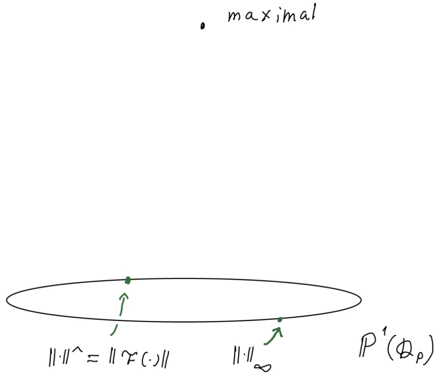
1. functional analysis:
   - Closed graph theorem, open mapping theorem, Baire's category theorem.
   - Other arguments special to the p-adic situation (studying the notion of strong irreducibility).
   - Those are used to study quotients of the completion of cyclic representations by maximal invariant norms (their universal unitary completion).
2. q-arithmetic in $C(\mathbb{Z}_p)$:
   - Appears in combinatorics, hypergeometric functions, modular forms, physics and more.
   - Recently in p-adic Hodge theory (B. Bhatt, M. Morrow, P. Scholze).
   - In our work: q-Mahler bases - a family of deformations of the Mahler basis $\{\binom{x}{n} \mid n \geq 0\}$.

q-arithmetic

- The q-analog notation:
  - The q-analog of $n \in \mathbb{N}$ is $[n]_q := \dfrac{q^n - 1}{q - 1} = 1 + q + ... + q^{n-1}$.
  - The q-analog of $n!$
  $$[n]_q! := [1]_q \cdot [2]_q \cdot ... \cdot [n]_q.$$
  - The q-analog of the binomial coefficients are
  $$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!}.$$
- When $q \in \mathbb{C}_p$ and $|q - 1|_p < 1$, $n \mapsto \begin{bmatrix} n \\ k \end{bmatrix}_q$ extends to $\begin{bmatrix} x \\ k \end{bmatrix}_q : \mathbb{Z}_p \to \mathbb{C}_p$.
- $\begin{bmatrix} x \\ 0 \end{bmatrix}_q, \begin{bmatrix} x \\ 1 \end{bmatrix}_q, \begin{bmatrix} x \\ 2 \end{bmatrix}_q, \ldots$ is an orthonormal basis of $C(\mathbb{Z}_p)$.
- Reduce to a rigidity statement for $\|\cdot\|_\infty$ on $C(\mathbb{Z}_p)$.
- Let $\|\cdot\|$ be a norm on $C(\mathbb{Z}_p)$ dominated by $\|\cdot\|_\infty$ and invariant under translations and multiplications by the smooth characters of $\mathbb{Z}_p$. Normalize it so that $\|\mathbf{1}\| = 1$.
- By studying the q-Mahler expansion of ζ^x ($\zeta^{p^n} = 1$) we show that for some q, the q-Mahler basis is an orthonormal basis for $\|\cdot\|$. Therefore, $\|\cdot\| = \|\cdot\|_\infty$.

Open problems

$d = 1$. Points $=$ equivalence classes of $\mathcal{H}_3(\mathbb{Q}_p)$-invariant norms on $\mathcal{S}(\mathbb{Q}_p)$.



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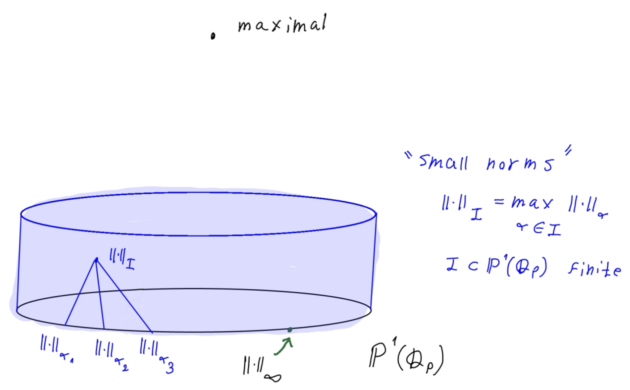

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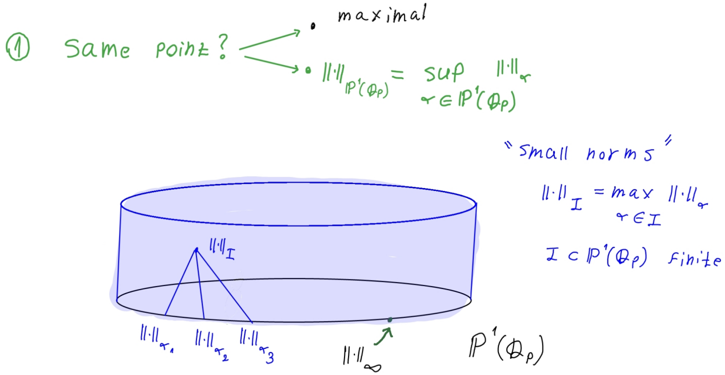

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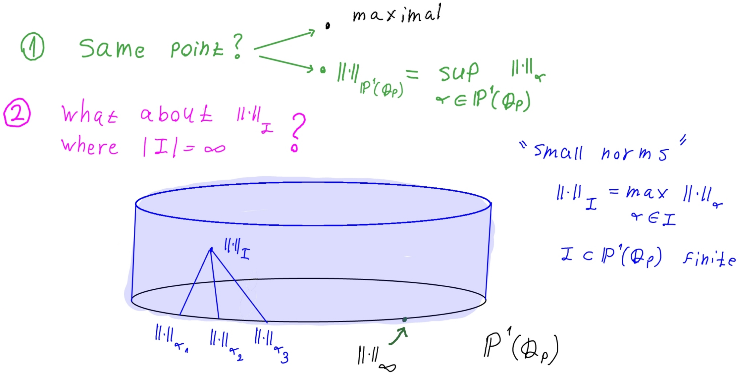

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